INTERPOLATION IN H^∞

BY

D. J. NEWMAN

Our purpose is to investigate sequences of points $\zeta_1, \zeta_2, \cdots, |\zeta_n| < 1$ for which the interpolation problem always has an analytic bounded solution. More precisely sequences such that, given any bounded sequence $a_1, a_2, \cdots$, there exists a bounded analytic (H^∞) function in $|z| < 1$ which satisfies $f(\zeta_n) = a_n$.

We call the class of such sequences $\{\zeta_n\}$, Q. Gleason[1] has recently proven that Q is non-null and has obtained certain sufficient conditions for a given sequence to be in Q. We shall give a necessary and sufficient condition for this, a condition which, in certain restricted cases, reduces to a very simple and elegant statement.

With the promise, then, that the corollaries will be prettier, we state our main theorem.

THEOREM 1. *A necessary and sufficient condition that* $\{\zeta_n\} \in Q$ *is*

1. $$\prod_{n \neq N; 1 \leqq n < \infty} \left| (1 - \bar{\zeta}_n \zeta_N)/(\zeta_N - \zeta_n) \right| \leqq M \quad (M \text{ independent of } N)$$

and

2. $$\sum (1 - |\zeta_n|) |G(\zeta_n)| < \infty, \text{ for all functions } G \in H^1.$$

Proof. In Banach space terms, we can phrase the statement $\{\zeta_n\} \in Q$ as follows: Define $\mathfrak{I}$: $H^\infty \to l^\infty$ (l^∞ is the space of bounded sequences) by

$$\mathfrak{I}(f(z)) = \{f(\zeta_1), f(\zeta_2), f(\zeta_3), \cdots\},$$

then

$$\{\zeta_n\} \in Q \rightleftarrows \mathfrak{I} \text{ is onto.}$$

It is in this framework which we give the proofs.

We first prove that 1. is necessary: for if $\mathfrak{I}$ is onto then, by the closed graph theorem,

$$\mathfrak{I}^{-1}: l^\infty \to \frac{H^\infty}{K} \quad (K \text{ is the kernel of } \mathfrak{I})$$

is a bounded operator.

Now consider any $f \in H^\infty \ni$

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[1] Communicated orally to the author by A. M. Gleason.

$$f(\zeta_n) = \delta_{nN}, \text{ then } \sup |f(z)|$$

$$= \sup \left| \prod_{n \neq N; n<R} \frac{1 - \bar{\zeta}_n z}{z - \zeta_n} \cdot f(z) \right| \geqq \prod_{n \neq N; n<R} \left| \frac{1 - \bar{\zeta}_n \zeta_N}{\zeta_N - \zeta_n} \right|,$$

but

$$\inf_{\text{such } f} \sup |f(z)| = \|\mathfrak{J}^{-1}(\delta_{1N}, \delta_{2N}, \delta_{3N}, \cdots)\| \leqq \|\mathfrak{J}^{-1}\| \cdot 1,$$

and so 1. follows with $M = \|\mathfrak{J}^{-1}\|$, by letting $R \to \infty$.

We now prove the remainder of the theorem which can now be stated. Given that (1) holds, (2) $\rightleftarrows \mathfrak{J}$ onto. First some lemmas:

LEMMA 1. $H^\infty = (L^1/H^1)^*$ (*by* L^1 *we mean* L^1 *on the circle* $|z| = 1$).

LEMMA 2. $l^\infty = l^{1*}$ (l^1 *is the space of abs. convergent series*).

LEMMA 3. *If* $T: l^1 \to L^1/H^1$ *is defined as taking* $(C_1, C_2, C_3, \cdots)$ *into* (*the residue class of*)

$$\frac{C_1}{z - \zeta_1} + \frac{C_2}{z - \zeta_2} + \frac{C_3}{z - \zeta_3} + \cdots$$

then T *is a bounded operator and* $\mathfrak{J} = T^*$.

Proofs. Lemma 1. Consider L^1/H^1, any functional on it can certainly be extended (by injection) to L^1 (we define $f(x) = f$(the residue class to which x belongs)) hence $(L^1/H^1)^* \subset L^\infty$, furthermore any L^∞ function which annihilates H^1 must be in H^∞. Finally it is all of H^∞ since H^∞ annihilates *only* H^1.

Lemma 2. See Banach [3].

Lemma 3. T is bounded, and the series converges, since the L^1/H^1 norm of $1/(z - \zeta_n)$ is

$$\leqq \int \left| \frac{1}{z - \zeta_n} - \frac{1}{z - \dfrac{1}{\bar{\zeta}_n}} \right| |dz| = 1$$

so that $\|T\| \leqq 1$. Consider T^*, this is defined such that $(C_1, C_2, \cdots) \cdot T^*(f(z)) = f(z) \cdot T(C_1, C_2, \cdots)$ but the right side $=$

$$\frac{1}{2\pi i} \int f(z) \left(\frac{C_1}{z - \zeta_1} + \cdots \right) dz = C_1 f(\zeta_1) + \cdots = (C_1, C_2, \cdots) \cdot \mathfrak{J}(f(z))$$

$\therefore\ T^* = \mathfrak{J}$.

In view of these lemmas we are able to take advantage of the following Theorem (2):

$T^*: C^* \to B^*$ is onto,

$\rightleftarrows T: B \to C$ has a bounded inverse [from $T(B)$ to B] in our case $\mathfrak{Z}$ onto $\rightleftarrows T: l^1 \to L^1/H^1$ has a bounded inverse and, since the L^1/H^1 norm of $C_1/(z-\zeta_1) + \cdots C_N/(z-\zeta_N)$ is

$$\inf_{g \in H'} \int_{|z|=1} \left| \frac{C_1}{z-\zeta_1} + \cdots \frac{C_N}{z-\zeta_N} - g(z) \right| |dz|,$$

the above condition is equivalent to

$$\exists M |C_1| + \cdots |C_N| \leqq M \int \left| \frac{C_1}{z-\zeta_1} + \frac{C_N}{z-\zeta_n} - g(z) \right| |dz|$$

for all $g \in H'$ and all $C_1, C_2, \cdots, C_N$.

If we now introduce $G(z)$ defined by

$$\prod_{R=1}^{N} \left(\frac{1-\bar{\zeta}_k z}{z-\zeta_k} \right) G(z) = \frac{C_1}{z-\zeta_1} + \cdots + \frac{C_n}{z-\zeta_n} - g(z)$$

then the requirement becomes

$$\sum_{n=1}^{N} \left(\prod_{k \neq n; 1 \leqq k \leqq N} \left| \frac{1-\bar{\zeta}_k \zeta_n}{\zeta_n - \zeta_k} \right| \right) (1 - |\zeta_n|^2) |G(\zeta_n)| \leqq M \int |G|,$$

for all $G \in H'$ but, since we are assuming (1) to hold, we have

$$1 \leqq \prod_{k \neq n; 1 \leqq k \leqq N} \left| \frac{1-\bar{\zeta}_k \zeta_n}{\zeta_n - \zeta_k} \right| \leqq M',$$

and so the above inequality is equivalent to

$$\sum_{n=1}^{N} (1 - |\zeta_n|) |G(\zeta_n)| \leqq M \int |G|,$$

but this in turn is equivalent to the statement

$$\sum_{n=1}^{\infty} (1 - |\zeta_n|) |G(\zeta_n)| < \infty \text{ for all } G \in H',$$

for if this statement holds then the transformation taking $G(z)$

$$G(z) \to ((1 - |\zeta_1|)G(\zeta_1), (1 - |\zeta_2|)G(\zeta_2), \cdots),$$

$[H^1 \to l^1]$ clearly has a closed graph and so

$$\sum (1 - |\zeta_n|) |G(\zeta_n)| \leqq M \int |G|$$

the converse is trivial.

This completes the proof of the theorem.

We now prove a sufficiency condition which assures us of plenty of sequences in Q.

THEOREM 2. *If ζ_n approach the boundary in exponential fashion, that is*

$$\frac{1-|\zeta_n|}{1-|\zeta_{n-1}|} < c < 1, \text{ then } \{\zeta_n\} \in Q.$$

Proof. We verify Conditions (1) and (2) of Theorem 1.

CONDITION 1. We can use the fact that

$$\left|\frac{\alpha-\beta}{1-\bar{\alpha}\beta}\right| \geqq \frac{|\alpha|-|\beta|}{1-|\alpha|\,|\beta|} \text{ if } |\alpha|,\ |\beta| < 1,$$

and conclude that

$$\prod_{n\neq N}\left|\frac{1-\bar{\zeta}_n\zeta_N}{\zeta_n-\zeta_N}\right| \leqq \prod_{n<N}\frac{1-|\zeta_n|\,|\zeta_N|}{|\zeta_N|-|\zeta_n|}\prod_{n>N}\frac{1-|\zeta_n|\,|\zeta_N|}{|\zeta_n|-|\zeta_N|}.$$

Now for $n>N$, by the hypothesis

$$1-|\zeta_n| \leqq c^{n-N}(1-|\zeta_N|),$$

and so

$$|\zeta_n|-|\zeta_N| \geqq (1-c^{n-N})(1-|\zeta_N|),$$

while

$$1-|\zeta_n|\,|\zeta_N| \leqq (1+c^{n-N})(1-|\zeta_N|),$$

$$\therefore \prod_{n>N} \leqq \prod_{k=1}^{\infty}\frac{1+c^k}{1-c^k}$$

for $n<N$; on the other hand,

$$1-|\zeta_N| \leqq c^{N-n}(1-|\zeta_n|)$$

so that

$$|\zeta_N|-|\zeta_n| \geqq (1-c^{N-n})(1-|\zeta_n|),$$

and

$$1-|\zeta_N|\,|\zeta_n| \leqq (1+c^{N-n})(1-|\zeta_n|),$$

$$\therefore \prod_{n<N} \leqq \prod_{k=1}^{\infty}\frac{1+c^k}{1-c^k}$$

and Condition (1) follows.

We remark that the same proof goes through, with minor changes, to insure

$$\prod_{n=1}^{\infty}\left|\frac{1-\bar{\zeta}_n z}{z-\zeta_n}\right| < M(\delta)$$

if z satisfies $|z-\zeta_n| > \delta(1-|\zeta_n|)$ for all n. This fact will prove of use later on.

CONDITION 2. Let $G\in H^1$ and suppose

$$G(z) = \sum_k \lambda_k z^k,$$

then

$$\sum (1-|\zeta_n|)|G(\zeta_n)| \leqq \sum_n (1-|\zeta_n|)\sum_k |\lambda_k|\,|\zeta_n|^k,$$
$$= \sum_k |\lambda_k| \sum_n (1-|\zeta_n|)|\zeta_n|^k.$$

Now let n_0 be the first n for which $1-|\zeta_n| < 1/(k+1)$ for $n<n_0$ we obtain

$$1-|\zeta_n| \geqq \frac{c^{n-n_0+1}}{k+1},$$

and since the function $(1-x)x^k$ is increasing for $1-x>1/(k+1)$

$$(1-|\zeta_n|)|\zeta_n|^k \leqq \frac{c^{n-n_0+1}}{k+1}\cdot\left(1-\frac{c^{n-n_0+1}}{k+1}.\right)^k$$
$$< \frac{2c^{n-n_0+1}}{k+1}\, e^{-c^{n-n_0+1}};$$

$$\therefore \sum_{n<n_0}(1-|\zeta_n|)|\zeta_n|^k < \frac{2}{k+1}\cdot\sum_{j=0}^{\infty} c^{-j}e^{-c^{-j}},$$

also we have

$$\sum_{n\geqq n_0}(1-|\zeta_n|)|\zeta_n|^k \leqq \sum_{n\geqq n_0}(1-|\zeta_n|),$$
$$\leqq \frac{1}{k+1}\sum_{n\geqq n_0} c^{n-n_0} = \frac{1}{k+1}\cdot\frac{1}{1-c}.$$

These two estimates together give

$$\sum_n (1-|\zeta_n|)|\zeta_n|^k < \frac{M}{k+1},$$

and so, inserting this in our initial inequality, we obtain

$$\sum (1-|\zeta_n|)|G(\zeta_n)| < M\sum_k \frac{|\lambda_k|}{k+1}.$$

It is known (3), however, that $\sum|\lambda_k|/(k+1)<\infty$ when $\sum\lambda_k z^k\in H^1$, and the proof is complete.

COROLLARY. *Q is non-null and in fact every infinite set of points inside the unit circle and having a limit point on the boundary has a subsequence which is* $\in Q$.

Since some of the inequalities used in proving Theorem 2 were quite crude the following may come as somewhat of a surprise.

THEOREM 3(2). *Let* ζ_n *consist of a sequence which converges to* $+1$ *in a nontangential manner*

$$\{\zeta_n\} \in Q \rightleftarrows \text{the set } \frac{1-\zeta_m}{1-\zeta_n}, \qquad m \neq n$$

does not have 1 *as a limit point.*

Proof. First of all we show that the above condition is implied by Condition (1), Theorem 1. Condition (1) gives

$$\left| \frac{1-\bar{\zeta}_n \zeta_N}{\zeta_n - \zeta_N} \right| < M, \qquad n \neq N.$$

so that

$$\frac{1-|\zeta_n|}{|\zeta_n - \zeta_N|} < M.$$

The hypothesis regarding nontangential approach states that

$$\frac{|1-\zeta_n|}{1-|\zeta_n|} < M'. \qquad \therefore \left| \frac{1-\zeta_n}{\zeta_n - \zeta_N} \right| < MM'$$

or

$$\left| \frac{1-\zeta_N}{1-\zeta_n} - 1 \right| > \frac{1}{MM'}$$

as required.

SUFFICIENCY. We first prove that the sequence can be partitioned into a finite number of subsequences each satisfying the hypothesis of Theorem 2. For consider the annulus

$$\frac{1}{2^{2n+1}} \leqq 1 - |z| < \frac{1}{2^{2n}};$$

this can only contain a bounded (with n) number of ζ's. If the number were unbounded then given ϵ, there would exist two different ζ's and an n such that

(2) This result can be extended to any finite number of boundary points.

$$|\zeta_i - \zeta_j| < \frac{\epsilon}{2^{2n+1}} < \epsilon |1 - \zeta_i|$$

and then 1 would be a limit point of $(1-\zeta_i)/(1-\zeta_j)$.

A similar remark holds for the annulus

$$\frac{1}{2^{2n+2}} \leqq 1 - |z| < \frac{1}{2^{2n+1}} \cdot$$

The predicted finite number of subsequences are then each formed by choosing one ζ from each of the annuli of the first type, or choosing one ζ from each of the annuli of the second type.

Condition (2) now follows immediately.

As for Condition (1) we use the remark made in the proof of Theorem 2. For let ζ be any element of our original sequence, and consider any one of our subsequences, call it $\beta_1, \beta_2, \cdots$. If we delete ζ from this sequence we have

$$|\zeta - \beta_n| > \frac{1}{M}(1 - |\beta_n|),$$

and so, by the aforementioned remark,

$$\prod \left| \frac{1 - \bar{\beta}_n \zeta}{\zeta - \beta_n} \right| < A.$$

This holding for each of the finite number of subsequences, Condition 1 clearly follows.

Theorem 3 is now complete.

In the interesting case of positive ζ_n's we have

COROLLARY. *If* $\zeta_n \geqq 0$ *and are arranged so as to be increasing, then a nasc for* $\{\zeta_n\} \in Q$ *is*

$$\frac{1 - \zeta_n}{1 - \zeta_{n-1}} < C < 1.$$

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASSACHUSETTS